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On Transforming a Tchebycheff System into a Complete Tchebycheff System

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Let $y_0,..., y_n$ be real-valued functions defined on a totally ordered set A. The system $\{y_0,..., y_n\}$ is said to be a Tchebycheff system (T-system) on A, provided that for every choice of points $t_0 < \cdots < t_n$ of A, the determinant $D(y_0, y_1,..., y_n/t_0, t_1,..., t_n) = \det\| y_i(t_j); i, j = 0,..., n\|$ is strictly positive, whereas if the determinant is merely nonnegative, the system is called a Weak Tchebycheff system (WT-system). If $\{y_0,..., y_k\}$ is a T-system on A for $k = 0,..., n$, then $\{y_0,..., y_n\}$ is called a Complete Tchebycheff system (CT-system) on A. The preceding definitions are consistent with Karlin and Studden [1].

M. G. Krein proved that if A is an open interval and $\{y_0,..., y_n\}$ is a T-system on A, then the linear span of the functions $y_0,..., y_n$ contains a CT-system thereon (cf. [2]). It seems that Krein never published his proof, and it was apparently Németh ([3], corollary on p. 310), who published the first proof of Krein's theorem. This theorem was recently generalized by Zielke [4], who showed that it holds for sets having "property (D)": a set A has property (D) if neither sup A nor inf A are contained in A, and for any two points of A there is a third point of A in between.

The purpose of this paper is to further generalize Krein's theorem, presenting at the same time a very short and elementary proof. Specifically, we shall prove the following assertion.

THEOREM. *Let A be a totally ordered set. If A has no smallest nor greatest element, then the linear span of every* T-*system on A contains a* CT-*system thereon.*

Proof. Assume first that A is a set of real numbers. Let $\{q_0,..., q_n\}$ be an ordered set of distinct points of A, and let $\{y_0,..., y_n\}$ be a T-system on A. Let $D = D(y_0,..., y_n/q_0,..., q_n)$, and define the functions r_i by means of the formula $r_i(t) = D(y_0,..., y_n/q_0,..., q_{i-1}, t, q_{i+1},..., q_n)$. The functions $r_0,..., r_n$ are clearly in the linear span of the functions $y_0,..., y_n$. Moreover, since

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$D(v_0, \ldots, v_n / q_0, \ldots, q_n) = D^{n+1} > 0$, proceeding as in [5], Lemma 2, we easily see that $\{v_0, \ldots, v_n\}$ is a T-system on A.

Let $\{b_m\}$; $m = 1, 2, \ldots$ be a strictly increasing sequence of points of A, all to the right of q_n, and converging to sup A, if A is bounded above, or to $+\infty$ if it is not. Define $z_n = (-1)^n v_0 + (-1)^{n-1} v_1 + \cdots + v_n$ and, for $i = 0, \ldots, n-1$, $z_i(\cdot, m) = v_i - c_i(m) z_n$, where $c_i(m) = v_i(b_m)/z_n(b_m)$. (Note that the functions $(-1)^{n-i} v_i$ are all strictly positive to the right of q_n; thus z_n also has this property.) It is clear that, for $i = 0, \ldots, n-1$, and $m = 1, 2, \ldots$, $z_i(b_m, m) = 0$; it is also quite obvious that $\{z_0(\cdot, m), \ldots, z_{n-1}(\cdot, m), z_n\}$ is a T-system on A.

Let A_m denote the set of points of A that precede b_m. We assert that if $m > m'$, then $\{z_0(\cdot, m), \ldots, z_{n-1}(\cdot, m)\}$ is a T-system on $A_{m'}$. In fact, let $t_0 < \cdots < t_{n-1}$ be points of $A_{m'}$. Since $t_{n-1} < b_m$, the conclusion follows by noting that $0 < D(z_0(\cdot, m), \ldots, z_{n-1}(\cdot, m), z_n / t_0, \ldots, t_{n-1}, b_m) = z_n(b_m) \cdot D(z_0(\cdot, m), \ldots, z_{n-1}(\cdot, m) / t_0, \ldots, t_{n-1})$, and that $z_n(b_m) > 0$.

By the definition of the coefficients $c_i(m)$, it is clear that they are bounded between 0 and 1. Thus, there exists a sequence $\{m_k\}$; $k = 1, 2, \ldots$, and numbers $c_0, \ldots, c_{n-1}$, such that $\lim_{k\to\infty} c_i(m_k) = c_i$; $i = 0, \ldots, n-1$. For $i = 0, \ldots, n-1$, let $z_i = v_i - c_i z_n$; clearly $\{z_0, \ldots, z_n\}$ is a T-system on A, and from the assertion proved in the preceding paragraph we readily see that $\{z_0, \ldots, z_{n-1}\}$ is a WT-system thereon.

Assume now that for some choice $t_0 < t_1 < \cdots < t_{n-1}$ of points of A,

$$D(z_0, \ldots, z_{n-1} / t_0, \ldots, t_{n-1}) = 0. \tag{1}$$

Let $t_n > t_{n-1}$ be a point of A. Since $D(z_0, \ldots, z_n / t_0, \ldots, t_n) > 0$, there is an integer i, $0 \leq i \leq n-1$, such that $D(z_0, \ldots, z_{n-1} / t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) > 0$. Define $z(t) = D(z_0, \ldots, z_n / t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n, t)$. Clearly z can be represented as a linear combination of the functions $z_0, \ldots, z_n$. The coefficient of z_n in this representation is $D(z_0, \ldots, z_{n-1} / t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, and thus strictly positive, whence $\{z_0, \ldots, z_{n-1}, z\}$ is a T-system on A. On the other hand, taking into consideration that $z(t_j) = 0$; $j = 0, \ldots, i-1, i+1, \ldots, n$, that $(-1)^{n-i} \cdot z(t_i) > 0$, and developing by the last row, we see from (1) that if t_0' is a point of A to the left of t_0, then $D(z_0, \ldots, z_{n-1}, z / t_0', t_0, \ldots, t_{n-1}) = (-1)^{n-i-1} z(t_i) \cdot D(z_0, \ldots, z_{n-1} / t_0', t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n-1}) \leq 0$ which is a contradiction. Repeating the above procedure for the system $\{z_0, \ldots, z_{n-1}\}$ and so on, the conclusion follows.

The preceding proof was carried out under the additional assumption that A is a set of real numbers. In order to prove the general case, it will suffice to show that if there is a T-system of at least two functions, defined on A, then there is a real-valued, strictly increasing function on A. Let $s_1 < s_2 < s_3$ be points of A. Let P denote the set of points of A to the left of s_3, and Q the set of points of A to the right of s_1. Assume first that the points q_i employed in the definition of the functions v_i above, are all to the left of s_1. If $t_0 < t_1$ are points of Q, it is clear that $0 < D(v_0, ..., v_n/q_0, ..., q_{n-2}, t_0, t_1) = D^{n-1} \cdot D(v_{n-1}, v_n/t_0, t_1)$.

Thus $\{v_{n-1}, v_n\}$ is a T-system on Q. In similar fashion, it is seen that $v_n > 0$ on Q, from which follows that $-v_{n-1}/v_n$ is strictly increasing thereon. Assuming now that the points q_i are all to the right of s_3, it can similarly be seen that v_1/v_0 is strictly increasing in P. Since P and Q have a common point, it is readily seen that there exists a strictly increasing, real-valued function h on A. The system $\{z_0, ..., z_n\}$ given by $z_i(t) = y_i[h^{-1}(t)]$ is a T-system on $h(A)$, and we have therefore transformed the problem into the one considered in the first case. Q.E.D.

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